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THE POTENTIAL GRADIENT
OF A PLANE SOURCE

by

A. V. Hershey

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THE POTENTIAL GRADIENT OF A PLANE SOURCE

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ABSTRACT

Direct integration leads to the potential gradient of a rectangular plate. Lines with equal spacing form a grid for integration over the infinite plane. A source distribution is approximated by pattern functions which are centered over each grid point. The pattern functions are based upon the sine quotient function. Fourier analysis replaces integration in physical space with integration in wave number space. The range of integration is limited to lines of finite length in wave number space. The computation of potential gradient is provided by subroutines.

INTRODUCTION

The flow around a ship is partly a radial flow from a source distribution and partly a circular flow around a vortex distribution. In either case the velocity is the integral of the product of an inverse square and the density of sources or vortices. The inverse square contributes a spike to the integrands and numerical integration is not feasible. A technique for integration through the spike may be found from an analysis of the analogous problem in the flow over a plane.

The potential and the potential gradient of a source or charge which is distributed over a plane have applications in hydrodynamics, electromagnetism, and gravitation. The field of an irregular three-dimensional source distribution can be constructed with source distributions over each of a stack of planes. The potential and the potential gradient at a field point near a plane can be computed by two methods.

In a direct method, the potential at a field point is the integral over the plane of the product of inverse distance and the source density, while the potential gradient is the integral over the plane of the product of an inverse square of distance and the source density. Wherever there is a discontinuity in source density the potential gradient is infinite.

In a Fourier method, the potential is expressed as a solution of Laplace's equation, and the solution is adjusted to fit the boundary conditions. The solution is the sum of products of complex exponential functions, and the source distribution is approximated as a Fourier series. A Fourier transform replaces integration in physical space with integration in wave number space. The time of computation depends upon the distance to the field point. There are two methods for the integration in wave number space. In the first method, Gauss integration multipliers are applied in a roving interval. The time increases with distance. In the second method, integration by parts leads to recurrence equations. The time is independent of distance.

DIRECT INTEGRATION

Let x, y, z be Cartesian coordinates in a right-handed coordinate system with x and y in the plane and with z above the plane. Let $\sigma(x, y)$ be the source density at a point with coordinates x, y . Then the potential is given by the equation

$$\varphi(x, y, z) = \iint \frac{\sigma(u, v)}{\sqrt{(x-u)^2 + (y-v)^2 + z^2}} du dv \quad (1)$$

where x, y, z are coordinates of a field point and u, v are coordinates of a source point.

RECTANGULAR PLATE

Let a rectangular plate be placed on the plane. Let the length of the plate be $2a$, and let the breadth of the plate be $2b$. Let the origin be at the center of the plate with the x -axis in the direction of length and with the y -axis in the direction of breadth.

The potential of a plate with unit source density is given by the equation

$$\varphi(x, y, z) = \int_{-b-y}^{+b-y} \int_{-a-x}^{+a-x} \frac{du dv}{\sqrt{u^2 + v^2 + z^2}} \quad (2)$$

Initial integration leads to inverse hyperbolic functions, then final integration is completed with integrations by parts.

The potential is given by the equation

$$\begin{aligned}
\varphi = & (a-x)\sinh^{-1}\frac{(b-y)}{\sqrt{(a-x)^2+z^2}} + (a-x)\sinh^{-1}\frac{(b+y)}{\sqrt{(a-x)^2+z^2}} \\
& + (a+x)\sinh^{-1}\frac{(b-y)}{\sqrt{(a+x)^2+z^2}} + (a+x)\sinh^{-1}\frac{(b+y)}{\sqrt{(a+x)^2+z^2}} \\
& + (b-y)\sinh^{-1}\frac{(a-x)}{\sqrt{(b-y)^2+z^2}} + (b-y)\sinh^{-1}\frac{(a+x)}{\sqrt{(b-y)^2+z^2}} \\
& + (b+y)\sinh^{-1}\frac{(a-x)}{\sqrt{(b+y)^2+z^2}} + (b+y)\sinh^{-1}\frac{(a+x)}{\sqrt{(b+y)^2+z^2}} \\
& - z \tan^{-1}\frac{(a-x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}} - z \tan^{-1}\frac{(a-x)(b+y)}{z\sqrt{(a-x)^2+(b+y)^2+z^2}} \\
& - z \tan^{-1}\frac{(a+x)(b-y)}{z\sqrt{(a+x)^2+(b-y)^2+z^2}} - z \tan^{-1}\frac{(a+x)(b+y)}{z\sqrt{(a+x)^2+(b+y)^2+z^2}} \quad (3)
\end{aligned}$$

Partial differentiation and cancellation of terms gives the components of the gradient of the potential.

Differentiation with respect to x leads to the equation

$$\begin{aligned}
-\frac{\partial\varphi}{\partial x} = & \sinh^{-1}\frac{(b-y)}{\sqrt{(a-x)^2+z^2}} + \sinh^{-1}\frac{(b+y)}{\sqrt{(a-x)^2+z^2}} \\
& - \sinh^{-1}\frac{(b-y)}{\sqrt{(a+x)^2+z^2}} - \sinh^{-1}\frac{(b+y)}{\sqrt{(a+x)^2+z^2}} \quad (4)
\end{aligned}$$

differentiation with respect to y leads to the equation

$$\begin{aligned}
-\frac{\partial\varphi}{\partial y} = & \sinh^{-1}\frac{(a-x)}{\sqrt{(b-y)^2+z^2}} + \sinh^{-1}\frac{(a+x)}{\sqrt{(b-y)^2+z^2}} \\
& - \sinh^{-1}\frac{(a-x)}{\sqrt{(b+y)^2+z^2}} - \sinh^{-1}\frac{(a+x)}{\sqrt{(b+y)^2+z^2}} \quad (5)
\end{aligned}$$

and differentiation with respect to z leads to the equation

$$\begin{aligned}
-\frac{\partial\varphi}{\partial z} = & \tan^{-1}\frac{(a-x)(b-y)}{z\sqrt{(a-x)^2+(b-y)^2+z^2}} + \tan^{-1}\frac{(a-x)(b+y)}{z\sqrt{(a-x)^2+(b+y)^2+z^2}} \\
& + \tan^{-1}\frac{(a+x)(b-y)}{z\sqrt{(a+x)^2+(b-y)^2+z^2}} + \tan^{-1}\frac{(a+x)(b+y)}{z\sqrt{(a+x)^2+(b+y)^2+z^2}} \quad (6)
\end{aligned}$$

These equations were published in a previous report¹. They are used in the following subroutines.

SUBROUTINE RPLTP (AA, AB, AX, AY, AZ, FP)

 FORTRAN SUBROUTINE FOR POTENTIAL OF RECTANGULAR PLATE

The half-length a of the plate is given in argument AA, the half-breadth b of the plate is given in argument AB, and the Cartesian coordinates x, y, z of the field point are given in arguments AX, AY, AZ. The potential of the plate is stored in function FP.

SUBROUTINE RPLTG (AA, AB, AX, AY, AZ, FX, FY, FZ)

 FORTRAN SUBROUTINE FOR POTENTIAL GRADIENT OF RECTANGULAR PLATE

The half-length a of the plate is given in argument AA, the half-breadth b of the plate is given in argument AB, and the Cartesian coordinates x, y, z of the field point are given in arguments AX, AY, AZ. The components of the gradient are stored in functions FX, FY, FZ.

NONUNIFORM PLATE

A major achievement was the integration when the source density was expressed as power polynomials in the surface coordinates. The computation of potential required a triplex of recurrence relations. The computation is provided by the subroutine RPLTM. The analysis has been presented in a previous report¹. That RPLTG and RPLTM give the same gradients has been verified by computation.

FOURIER ANALYSIS

The Fourier transform for a function on a plane is given by the equations

$$A(\alpha, \beta) = \frac{1}{4\pi^2} \int \int F(x, y) e^{-i(\alpha x + \beta y)} dx dy \quad (7)$$

$$F(x, y) = \int \int A(\alpha, \beta) e^{i(\alpha x + \beta y)} d\alpha d\beta \quad (8)$$

where x, y are Cartesian coordinates in physical space, α, β are Cartesian coordinates in wave number space, $F(x, y)$ is a function in physical space, and $A(\alpha, \beta)$ is the Fourier amplitude in wave number space. A potential $\varphi(x, y, z)$ of a source distribution on the plane can be constructed with the equation

$$\varphi(x, y, z) = \int \int A(\alpha, \beta) e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)} d\alpha d\beta \quad (9)$$

which gives a solution of Laplace's equation wherever $z \neq 0$. The vertical component of the gradient at $z = 0$ is given by the equation

$$-\frac{\partial \varphi}{\partial z} = \pm \int \int \sqrt{\alpha^2 + \beta^2} A(\alpha, \beta) e^{i(\alpha x + \beta y)} d\alpha d\beta \quad (z \rightarrow 0) \quad (10)$$

where the sign is + if $z > 0$ and the sign is - if $z < 0$. In accordance with the Gauss theorem the difference in $-\partial \varphi / \partial z$ on opposite sides of the plane is $4\pi\sigma(x, y)$ where $\sigma(x, y)$ is the surface source density at the plane. Thus the Fourier amplitude for an

arbitrary distribution of source density is given by the equation

$$A(\alpha, \beta) = \frac{1}{2\pi\sqrt{\alpha^2 + \beta^2}} \iint \sigma(x, y) e^{-i(\alpha x + \beta y)} dx dy \quad (11)$$

The range of integration of density is the infinite plane.

RECTANGULAR PLATE

Let a rectangular plate be placed on the plane. Let the length of the plate be $2a$, and let the breadth of the plate be $2b$. Let the origin be at the center of the plate with the x -axis in the direction of length and with the y -axis in the direction of breadth. Let u, v be the coordinates of a source point on the plate and let x, y, z be the coordinates of a field point above the plate.

The potential of a plate with unit source density is given by the equation

$$\varphi(x, y, z) = \frac{1}{2\pi} \iiint \frac{e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha(x-u) + \beta(y-v))}}{\sqrt{\alpha^2 + \beta^2}} d\alpha d\beta du dv \quad (12)$$

Completion of the integration leads to the equation

$$\varphi(x, y, z) = \frac{2}{\pi} \iint \frac{\sin(\alpha a)}{\alpha} \frac{\sin(\beta b)}{\beta} \frac{e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)}}{\sqrt{\alpha^2 + \beta^2}} d\alpha d\beta \quad (13)$$

What is interesting about this equation is that the distribution of source density over a finite plate introduces the factors $\sin(\alpha a)/\alpha$ and $\sin(\beta b)/\beta$ into the Fourier amplitude. The factors concentrate the spectrum of the Fourier integral into bands which are centered with respect to the line spectrum of a trigonometric polynomial.

INTERPOLATION

The coefficient function for Lagrange interpolation in an infinite set of equally spaced data is just the infinite product for the sine quotient function in the equation

$$\frac{\sin u}{u} = \prod_{k=1}^{\infty} \left(1 - \frac{u^2}{k^2\pi^2}\right) \quad (14)$$

It is equal to unity at the origin and is equal to zero wherever else u is a multiple of π . It is analytic and diminishes with distance from the origin.

An arbitrary function $f(u)$ is expressed in terms of coefficient functions with centers at $k\pi$ by the equation

$$f(u) = \sum_{k=-\infty}^{k=+\infty} f(u_k) \frac{\sin(u - k\pi)}{u - k\pi} \quad (15)$$

where $f(u_k)$ is the discrete value of $f(u)$ at the k th node.

The product of functions which are centered at $k\pi$ and $m\pi$ is given by the function

$$\frac{\sin(u - k\pi)}{(u - k\pi)} \frac{\sin(u - m\pi)}{(u - m\pi)} \quad (16)$$

Resolution into partial fractions and application of the addition theorem for

trigonometric functions leads to the equation

$$\int_{-\infty}^{+\infty} \frac{\sin(u-k\pi)}{(u-k\pi)} \frac{\sin(u-m\pi)}{(u-m\pi)} du = \frac{(-1)^{k+m}}{(k-m)\pi} \left[\int_{-\infty}^{+\infty} \frac{\sin^2(u-k\pi)}{u-k\pi} du - \int_{-\infty}^{+\infty} \frac{\sin^2(u-m\pi)}{u-m\pi} du \right] \quad (17)$$

Inasmuch as the integrands are odd their integrals are zero and the coefficient functions are orthogonal with respect to integration. They may be used as the basis for approximations by least squares.

An important property is expressed by the equation

$$\int_{-\infty}^{+\infty} \frac{\sin au}{u} du = \begin{cases} -\pi & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ +\pi & \text{if } a > 0 \end{cases} \quad (18)$$

where a is a constant of arbitrary magnitude. Integration by parts leads to the equation

$$\int_{-\infty}^{+\infty} \frac{\sin^2 u}{u^2} du = \int_{-\infty}^{+\infty} \frac{\sin 2u}{u} du = \pi \quad (19)$$

which gives the normalization factor for least squares approximation.

SOURCE PATTERN

A universal source pattern for the infinite plane is given by the equation

$$F(x, y) = \frac{\sin\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi y}{2b}\right)}{\left(\frac{\pi x}{2a}\right) \left(\frac{\pi y}{2b}\right)} \quad (20)$$

The function $F(x, y)$ is unity at the origin and is zero on all equally spaced nodal lines which do not pass through the origin. The spacing between nodal lines is $2a$ in the x -direction and is $2b$ in the y -direction. The function is analytic and diminishes with distance from the origin. Application of the Fourier transform and use of the Euler theorem introduces the difference of two cosines and introduces the sum of two sines into the integrand of the Fourier amplitude. The difference between the cosines is zero at the origin and is an even function otherwise. The difference between the cosines is eliminated by integration. There remains the sum of sines as expressed by the equation

$$A(\alpha, \beta) = \frac{1}{16\pi^2} \int \int \frac{\sin(\frac{\pi}{2a}-\alpha)x + \sin(\frac{\pi}{2a}+\alpha)x}{\left(\frac{\pi x}{2a}\right)} \cdot \frac{\sin(\frac{\pi}{2b}-\beta)y + \sin(\frac{\pi}{2b}+\beta)y}{\left(\frac{\pi y}{2b}\right)} dx dy \quad (21)$$

The sines cancel each other during integration wherever α, β are outside a rectangle in wave number space. The interior of the rectangle is located where α, β satisfy the inequalities

$$-\frac{\pi}{2a} \leq \alpha \leq +\frac{\pi}{2a} \qquad -\frac{\pi}{2b} \leq \beta \leq +\frac{\pi}{2b} \quad (22)$$

Thus the function $F(x, y)$ is given by the equation

$$F(x, y) = \frac{ab}{\pi^2} \int_{-\frac{\pi}{2b}}^{+\frac{\pi}{2b}} \int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}} e^{i(\alpha x + \beta y)} d\alpha d\beta = \sigma(x, y) \quad (23)$$

and the potential $\varphi(x, y, z)$ is given by the equation

$$\varphi(x, y, z) = \frac{2ab}{\pi} \int_{-\frac{\pi}{2b}}^{+\frac{\pi}{2b}} \int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}} \frac{e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)}}{\sqrt{\alpha^2 + \beta^2}} d\alpha d\beta \quad (24)$$

Further integration is possible after Cartesian coordinates α, β are replaced by polar coordinates κ, θ . The potential is expressed by the equation

$$\varphi(x, y, z) = \frac{2ab}{\pi} \int \int e^{-\kappa|z| + i\kappa(x\cos\theta + y\sin\theta)} d\kappa d\theta \quad (25)$$

Then integration with respect to κ replaces surface integration throughout the interior of the rectangle with line integration along the perimeter of the rectangle.

The three components of the gradient are given by the same formulations except for a weighting function w . Thus for the components

$$-\frac{\partial\varphi}{\partial x} \quad -\frac{\partial\varphi}{\partial y} \quad -\frac{\partial\varphi}{\partial z} \quad (26)$$

the values of w are

$$-i\cos\theta \quad -i\sin\theta \quad +1 \quad (27)$$

Then each component is the sum

$$A + B \quad (28)$$

of two integrals where A is the integral along vertical sides of the rectangle and B is the integral along horizontal sides of the rectangle.

For the vertical sides the variables are related in accordance with the equations

$$t = \tan\theta \quad (29)$$

$$\sin\theta = \frac{t}{\sqrt{1+t^2}} \quad \cos\theta = \frac{1}{\sqrt{1+t^2}} \quad d\theta = +\frac{dt}{1+t^2} \quad (30)$$

$$\alpha = \kappa \cos\theta = \frac{\pi}{2a} \quad (31)$$

Let a parameter δ be defined by the equation

$$\delta = \frac{\pi}{2a} \{ |z|\sqrt{1+t^2} - i(x+yt) \} \quad (32)$$

Then the integral A is given by the equation

$$A = \pi \frac{b}{a} \int_{-\frac{a}{b}}^{+\frac{a}{b}} \frac{1 - (1+\delta)e^{-\delta}}{\delta^2} w dt \quad (33)$$

for integration along both vertical sides.

For the horizontal sides the variables are related in accordance with the equations

$$t = \cot \theta \quad (34)$$

$$\sin \theta = \frac{1}{\sqrt{1+t^2}} \quad \cos \theta = \frac{t}{\sqrt{1+t^2}} \quad d\theta = -\frac{dt}{1+t^2} \quad (35)$$

$$\beta = \kappa \sin \theta = \frac{\pi}{2b} \quad (36)$$

Let the parameter δ be defined by the equation

$$\delta = \frac{\pi}{2b} \{ |z| \sqrt{1+t^2} - i(xt+y) \} \quad (37)$$

Then the integral B is given by the equation

$$B = \pi \frac{a}{b} \int_{-\frac{b}{a}}^{+\frac{b}{a}} \frac{1 - (1+\delta)e^{-\delta}}{\delta^2} w dt \quad (38)$$

for integration along both horizontal sides.

Wherever $|\delta| \leq 1$ the integrand is replaced by the infinite series in the equation

$$\frac{1 - (1+\delta)e^{-\delta}}{\delta^2} = \sum_{k=0}^{\infty} (k+1) \frac{(-\delta)^k}{(k+2)!} \quad (39)$$

Summation of this series is continued until there is no change in the sum.

GAUSS INTEGRATION

The integration is completed with the aid of 16-point Gaussian integration multipliers. Each interval of integration is subdivided into subintervals and the Gaussian integration is applied progressively to successive subintervals. The numbers of subintervals for the integration of A and B are the integral parts of the expressions

$$2 + \frac{|y|}{7b} \quad 2 + \frac{|x|}{7a} \quad (40)$$

Then the Gaussian integration is accurate for any x or y . For large values of x, y the time for computation is nearly proportional to the sum of $|x|$ and $|y|$.

Formulae for the Taylor series expansion of the source pattern at nodal points in a finite grid were used with subroutine RPLTM to obtain gradients at several grid sizes. Extrapolation to infinite grid size confirmed the line integration in wave number space.

The components of the gradient of potential are given by the following subroutine.

SUBROUTINE RPTNG (AA, AB, AX, AY, AZ, FX, FY, FZ)

 FORTRAN SUBROUTINE FOR POTENTIAL GRADIENT OF RECTANGULAR PATTERN

The half-interval a of the pattern is given in argument AA, the half-interval b of the pattern is given in argument AB, and the Cartesian coordinates x, y, z of the field point are given in arguments AX, AY, AZ. The components of the gradient are computed by Gauss integration and are stored in the functions FX, FY, FZ.

INTEGRATION BY PARTS

For the integration by parts, the quadratic formula is used to solve the equations for δ to give t as a function of δ .

For the vertical sides of the rectangle the variable t is given by the equation

$$t = \frac{iy(\frac{2a\delta}{\pi} + ix) \pm |z|\sqrt{(\frac{2a\delta}{\pi} + ix)^2 - (y^2 + z^2)}}{y^2 + z^2} \quad (41)$$

Substitution and cancellation lead to the equation

$$\sqrt{1 + t^2} = \frac{|z|(\frac{2a\delta}{\pi} + ix) \pm iy\sqrt{(\frac{2a\delta}{\pi} + ix)^2 - (y^2 + z^2)}}{y^2 + z^2} \quad (42)$$

and differentiation leads to the equation

$$\frac{dt}{d\delta} = \frac{\frac{2a}{\pi}}{(y^2 + z^2)} \left[\frac{iy\sqrt{(\frac{2a\delta}{\pi} + ix)^2 - (y^2 + z^2)} \pm |z|(\frac{2a\delta}{\pi} + ix)}{\sqrt{(\frac{2a\delta}{\pi} + ix)^2 - (y^2 + z^2)}} \right] \quad (43)$$

The \pm sign in the equations is determined by substitution of the limits of integration for t . The sign is $+$ for positive t and is $-$ for negative t .

Substitution in the equation for A replaces integration with respect to t with integration with respect to δ . The natural path of integration in the t -plane is the real axis, whereas the natural path of integration in the δ -plane is an hyperbola with vertex where δ is given by the equation

$$\delta = \frac{\pi}{2a} \{|z| - ix\} \quad (44)$$

and with axis in the direction of the positive real axis. There is a singularity on each side of the imaginary axis. Near the singularity on the positive side δ is given by the equation

$$\delta = \frac{\pi}{2a} \{\sqrt{y^2 + z^2} - ix\} + \epsilon \quad (45)$$

Then the limiting values for functions at the singularity are given by the equations

$$t = \frac{iy}{\sqrt{y^2 + z^2}} \quad \sqrt{1 + t^2} = \frac{|z|}{\sqrt{y^2 + z^2}} \quad (46)$$

and by the equation

$$\sqrt{\epsilon} \frac{dt}{d\delta} \rightarrow \frac{|z|\sqrt{\frac{a}{\pi}}}{(y^2 + z^2)^{\frac{3}{4}}} \quad (\epsilon \rightarrow 0) \quad (47)$$

The hyperbolic path in the δ -plane is deformed into two straight lines which run from the positive singularity to the limits of integration where $t = \pm a/b$.

For the horizontal sides of the rectangle the variable t is given by the equation

$$t = \frac{ix(\frac{2b\delta}{\pi} + iy) \pm |z|\sqrt{(\frac{2b\delta}{\pi} + iy)^2 - (x^2 + z^2)}}{x^2 + z^2} \quad (48)$$

Substitution and cancellation lead to the equation

$$\sqrt{1+t^2} = \frac{|z|(\frac{2b\delta}{\pi} + iy) \pm ix\sqrt{(\frac{2b\delta}{\pi} + iy)^2 - (x^2 + z^2)}}{x^2 + z^2} \quad (49)$$

and differentiation leads to the equation

$$\frac{dt}{d\delta} = \frac{\frac{2b}{\pi}}{(x^2 + z^2)} \left[\frac{ix\sqrt{(\frac{2b\delta}{\pi} + iy)^2 - (x^2 + z^2)} \pm |z|(\frac{2b\delta}{\pi} + iy)}{\sqrt{(\frac{2b\delta}{\pi} + iy)^2 - (x^2 + z^2)}} \right] \quad (50)$$

The \pm sign in the equations is determined by substitution of the limits of integration for t . The sign is $+$ for positive t and is $-$ for negative t .

Substitution in the equation for B replaces integration with respect to t with integration with respect to δ . The natural path of integration in the t -plane is the real axis, whereas the natural path of integration in the δ -plane is an hyperbola with vertex where δ is given by the equation

$$\delta = \frac{\pi}{2b} \{|z| - iy\} \quad (51)$$

and with axis in the direction of the positive real axis. There is a singularity on each side of the imaginary axis. Near the singularity on the positive side δ is given by the equation

$$\delta = \frac{\pi}{2b} \{\sqrt{x^2 + z^2} - iy\} + \epsilon \quad (52)$$

Then the limiting values for functions at the singularity are given by the equations

$$t = \frac{ix}{\sqrt{x^2 + z^2}} \quad \sqrt{1+t^2} = \frac{|z|}{\sqrt{x^2 + z^2}} \quad (53)$$

and by the equation

$$\sqrt{\epsilon} \frac{dt}{d\delta} \rightarrow \frac{|z|\sqrt{\frac{b}{\pi}}}{(x^2 + z^2)^{\frac{3}{4}}} \quad (\epsilon \rightarrow 0) \quad (54)$$

The hyperbolic path in the δ -plane is deformed into two straight lines which run from the positive singularity to the limits of integration where $t = \pm b/a$.

The weighted derivative

$$w \frac{dt}{d\delta} \quad (55)$$

is expressed as a power series expansion by 11-point Lagrange interpolation. Let δ be given by the equation

$$\delta = \eta + \epsilon \quad (56)$$

where η is a center of expansion and ϵ is the argument of expansion. Inasmuch as the paths of integration are straight lines in the ϵ -plane, all values of ϵ in the range of interpolation are given by the equation

$$\epsilon = u\epsilon_M \quad (-1 \leq u \leq +1) \quad (57)$$

where ϵ_M is the upper limit of interpolation and u is the variable of interpolation. The limit ϵ_M can be cancelled out of the rational expression for the complex Lagrange

coefficient and the derivative can be expanded as a power series in u . In a preliminary computation discrete values for u with Chebyshev spacing and discrete coefficients were computed for use in a subroutine. Thus the derivatives are expressed as power polynomials in the ratio

$$u = \frac{\epsilon}{\epsilon_M} \quad (58)$$

while division by ϵ_M is deferred until the integration by parts.

There are three stages of expansion. In the first stage, η coincides with the singularity and derivatives are expanded as power polynomials in $\epsilon^{1/2}$. The range of interpolation is extended into a negative range in order to obtain data for central interpolation. The extension in the $\epsilon^{1/2}$ -plane corresponds to a fold in the ϵ -plane. The interval of integration is the upper half of the range of interpolation. In the later stages of expansion, η is moved further out along the path of integration and the derivatives are expanded as power polynomials in ϵ . The interval of integration is the full range of interpolation.

There are tight limitations on the range of interpolation. The radius of expansion near the first singularity must be a small fraction of the distance from the first singularity to the second singularity. The radius of expansion near the second center of expansion must be a small fraction of the distance from the center of expansion to either singularity.

The integration of each series in powers of $\epsilon^{1/2}$ is completed with a descending recurrence. Required for the integration are integrals which are generated by the recurrence

$$\int_0^\epsilon \epsilon^{m-1} e^{-\delta} d\epsilon = \frac{\epsilon^m}{m} e^{-\delta} + \frac{1}{m} \int_0^\epsilon \epsilon^m e^{-\delta} d\epsilon \quad (59)$$

The iteration is started with an initial approximation for $m = 32$. Completion of integration is given by integrals which are obtained by the recurrence

$$\begin{aligned} \int_0^\epsilon \epsilon^{m-1} \frac{1 - (1 + \delta)e^{-\delta}}{\delta^2} d\epsilon &= \frac{\epsilon^m}{m\eta} \frac{1 - (1 + \delta)e^{-\delta}}{\delta} - \frac{1}{m\eta} \int_0^\epsilon \epsilon^m e^{-\delta} d\epsilon \\ &\quad - \frac{m-1}{m\eta} \int_0^\epsilon \epsilon^m \frac{1 - (1 + \delta)e^{-\delta}}{\delta^2} d\epsilon \end{aligned} \quad (60)$$

where m is alternately half an odd integer and half an even integer. The iteration is started with an initial approximation for $m = 32$. Stability of the recurrence is achieved by a limitation of $|\epsilon|$ to $\frac{1}{\sqrt{2}}|\eta|$.

The integration of each series in powers of ϵ is cycled in either direction. Required for the integration are integrals which are generated by the ascending recurrence

$$\int_0^\epsilon \epsilon^m e^{-\delta} d\epsilon = -\epsilon^m e^{-\delta} + m \int_0^\epsilon \epsilon^{m-1} e^{-\delta} d\epsilon \quad (61)$$

The iteration is started with the special integral in the equation

$$\int_0^\epsilon \epsilon e^{-\delta} d\epsilon = e^{-\eta} - (1 + \epsilon)e^{-\delta} \quad (62)$$

and the iteration is continued until $m = 64$. If $|\epsilon| \leq 17$ then the iteration is cycled in descending order with an initial approximation. Completion of integration is given by

integrals which are generated by the ascending recurrence

$$\begin{aligned} \int_0^\epsilon \epsilon^m \frac{1 - (1 + \delta)e^{-\delta}}{\delta^2} d\epsilon &= \frac{\epsilon^m}{m-1} \frac{1 - (1 + \delta)e^{-\delta}}{\delta} - \frac{1}{m-1} \int_0^\epsilon \epsilon^m e^{-\delta} d\epsilon \\ &- \frac{m\eta}{m-1} \int_0^\epsilon \epsilon^{m-1} \frac{1 - (1 + \delta)e^{-\delta}}{\delta^2} d\epsilon \end{aligned} \quad (63)$$

where m is a sequence of integers. The iteration is started with the aid of the special integrals in the equations

$$\int_0^\delta \frac{1 - (1 + u)e^{-u}}{u^2} du = 1 - \frac{1 - e^{-\delta}}{\delta} = - \sum_{k=0}^{\infty} \frac{(-\delta)^{k+1}}{(k+2)!} \quad (64)$$

$$\int_0^\delta \frac{1 - (1 + u)e^{-u}}{u} du = -1 + e^{-\delta} + \gamma + \log \delta - Ei(-\delta) = \sum_{k=0}^{\infty} \frac{(-\delta)^{k+2}}{(k+2)^2 k!} \quad (65)$$

$$\int_0^\delta \{1 - (1 + u)e^{-u}\} du = -2 + \delta + (2 + \delta)e^{-\delta} = - \sum_{k=0}^{\infty} \frac{(-\delta)^{k+3}}{(k+3)(k+2)k!} \quad (66)$$

and the iteration is continued for 11 cycles. If $|\epsilon| \leq \frac{1}{2}|\eta|$ the iteration is cycled in descending order with an initial approximation.

In the actual program ϵ is replaced by ϵ/ϵ_M , which gives integrals ready for use with the Lagrange coefficients, and scales the integrals so that their exponents do not exceed the tight limitation on the exponents in IBM computers. The recurrence equations can be verified easily by differentiation.

Although the integration by parts is accurate everywhere that $|z|$ is not small, the integration by parts breaks down when $y^2 + z^2$ is zero along the vertical sides and breaks down when $x^2 + z^2$ is zero along the horizontal sides. Under such circumstances the program falls back upon Gauss integration.

For large values of x, y the time for computation is independent of $|x|$ or $|y|$. The components of the gradient of the potential are given by the following subroutine.

SUBROUTINE RPTNX (AA, AB, AX, AY, AZ, FX, FY, FZ)

 FORTRAN SUBROUTINE FOR POTENTIAL GRADIENT OF RECTANGULAR PATTERN

The half-interval a of the pattern is given in argument AA, the half-interval b of the pattern is given in argument AB, and the Cartesian coordinates x, y, z of the field point are given in arguments AX, AY, AZ. The components of the gradient are computed with integration by parts and are stored in functions FX, FY, FZ.

GRADIENT OVER GRID

Exploratory computations have given the gradient of potential on a vertical line through each grid point of the source pattern. The gradient has a direction which passes through the center and has a magnitude which varies with height. Far from the plane the magnitude approaches the full value for the inverse square law, while near the plane the magnitude is diminished by shielding by the local structure of the source pattern. As distance diminishes toward the center the inverse square of distance approaches ∞ , while the gradient approaches 2π .

The potential is given by the equation

$$\varphi(x, y, z) = \frac{2ab}{\pi} \int_{-\frac{\pi}{2b}}^{+\frac{\pi}{2b}} \int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}} \frac{e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)}}{\sqrt{\alpha^2 + \beta^2}} d\alpha d\beta \quad (67)$$

Combinations of the partial derivatives replace the integrand with an exact differential which can be integrated at once. Results of integration are given by the equation

$$i \left(|z| \frac{\partial \varphi}{\partial x} - x \frac{\partial \varphi}{\partial z} \right) = \frac{2ab}{\pi} \int_{-\frac{\pi}{2b}}^{+\frac{\pi}{2b}} \left[e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)} \right]_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}} d\beta \quad (68)$$

The integral is zero if x is a multiple of $2a$. Results of integration are given by the equation

$$i \left(|z| \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial z} \right) = \frac{2ab}{\pi} \int_{-\frac{\pi}{2a}}^{+\frac{\pi}{2a}} \left[e^{-\sqrt{\alpha^2 + \beta^2}|z| + i(\alpha x + \beta y)} \right]_{-\frac{\pi}{2b}}^{+\frac{\pi}{2b}} d\alpha \quad (69)$$

The integral is zero if y is a multiple of $2b$. Thus the potential gradient has a radial direction over grid points where x, y are multiples of $2a, 2b$. The integration of the inverse square over the surface of the pattern is just the gradient of a point source when the field point is over a grid point.

RECTILINEAR LINE

When the plane contracts to a line, the potential and the potential gradient become symmetric with respect to the line. The position of a field point is located by the cylindrical polar coordinates r, ϕ, z , where r is the distance from the line, ϕ is the azimuth angle, and z is the zenith distance along the line. The Fourier transform for a function on the line is given by the equations

$$A(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(z) e^{-i\kappa z} dz \quad (70)$$

$$f(z) = \int_{-\infty}^{+\infty} A(\kappa) e^{i\kappa z} d\kappa \quad (71)$$

where z is the coordinate in physical space and κ is the coordinate in wave number space.

A universal source pattern for the infinite line is defined by the equation

$$f(z) = \frac{\sin\left(\frac{\pi z}{2c}\right)}{\left(\frac{\pi z}{2c}\right)} \quad (72)$$

where the spacing between nodal points is $2c$. Application of the Fourier transform and use of the Euler theorem introduces the difference of two cosines and the sum of two sines into the integrand of the Fourier amplitude. The difference of cosines is an even function of z , and the sum of sines is an odd function of z . Both functions are divided by the odd function z . The cosines are cancelled by the integration. There

remains the sum of sines as expressed by the equation

$$A(\kappa) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\sin(\frac{\pi}{2c} - \kappa)z + \sin(\frac{\pi}{2c} + \kappa)z}{\left(\frac{\pi z}{2c}\right)} dz \quad (73)$$

Then the sines cancel each other whenever κ is outside the range

$$-\frac{\pi}{2c} \leq \kappa \leq +\frac{\pi}{2c} \quad (74)$$

and the amplitude is given by the equation

$$A(\kappa) = \frac{c}{\pi} \quad (75)$$

whenever κ is inside the range. Thus the function $f(z)$ is given by the equation

$$f(z) = \frac{c}{\pi} \int_{-\frac{\pi}{2c}}^{+\frac{\pi}{2c}} e^{i\kappa z} d\kappa = \frac{2c}{\pi} \int_0^{\frac{\pi}{2c}} \cos(\kappa z) d\kappa \quad (76)$$

where the range of integration in wave number space is finite.

The Laplacian in cylindrical polar coordinates is given by the equation

$$\nabla \cdot \nabla \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2} \quad (77)$$

The solutions of Laplace's equation are given in many texts. In this case the appropriate solution contains the modified Bessel function of the second kind, because this function diminishes to zero with increasing argument^{2,3}

The potential is given by the equation

$$\varphi(r, z) = \frac{2c}{\pi} \int_{-\frac{\pi}{2c}}^{+\frac{\pi}{2c}} e^{i\kappa z} K_0(\kappa r) d\kappa = \frac{4c}{\pi} \int_0^{\frac{\pi}{2c}} \cos(\kappa z) K_0(\kappa r) d\kappa \quad (78)$$

where $K_0(\kappa r)$ is the Bessel function of order zero. The Bessel functions satisfy the equation

$$\frac{d}{dr} K_0(\kappa r) = -\kappa K_1(\kappa r) \quad (79)$$

where $K_1(\kappa r)$ is the Bessel function of order one. The Bessel functions satisfy the limits

$$K_0(\kappa r) \rightarrow -\log(\kappa r) \quad K_1(\kappa r) \rightarrow \frac{1}{\kappa r} \quad (r \rightarrow 0) \quad (80)$$

at small radius. Thus the potential satisfies the limit equation

$$-2\pi r \frac{\partial \varphi}{\partial r} = 4\pi f(z) \quad (r \rightarrow 0) \quad (81)$$

where $f(z)$ is the linear density $\sigma(z)$ in accordance with the Gauss theorem.

The components of the gradient of potential are given by the equations

$$-\frac{\partial \varphi}{\partial r} = + \frac{2c}{\pi} \int_{-\frac{\pi}{2c}}^{+\frac{\pi}{2c}} \kappa e^{i\kappa z} K_1(\kappa r) d\kappa = + \frac{4c}{\pi} \int_0^{\frac{\pi}{2c}} \kappa \cos(\kappa z) K_1(\kappa r) d\kappa \quad (82)$$

$$-\frac{\partial \varphi}{\partial z} = - \frac{2ic}{\pi} \int_{-\frac{\pi}{2c}}^{+\frac{\pi}{2c}} \kappa e^{i\kappa z} K_0(\kappa r) d\kappa = + \frac{4c}{\pi} \int_0^{\frac{\pi}{2c}} \kappa \sin(\kappa z) K_0(\kappa r) d\kappa \quad (83)$$

Inasmuch as these integrals are limited to a finite range, they are evaluated with Gauss integration multipliers.

The potential gradient of a rectilinear source pattern is given by the following subroutines.

SUBROUTINE BSSLK (MO, AZ, IN, FK)

 FORTRAN SUBROUTINE FOR MODIFIED BESSEL FUNCTION OF INTEGRAL ORDER

The mode of operation is given in MO. The real and imaginary parts of the argument z are given in array AZ, and the integer order n is given in IN. The modified Bessel function of the second kind is computed by series expansion, rational approximation, and recurrence relations. If MO = 0, the real and imaginary parts of the function $K_n(z)$ are stored in array FK. If MO = 1, the real and imaginary parts of the function $e^z K_n(z)$ are stored in array FK.

SUBROUTINE LPTNG (AC, AR, AZ, FR, FZ)

 FORTRAN SUBROUTINE FOR POTENTIAL GRADIENT OF LINEAR PATTERN

The half-interval c of the pattern is given in argument AC, and the cylindrical coordinates r, z of the field point are given in arguments AR, AZ. The components of the gradient are computed by Gauss integration and are stored in the functions FR, FZ.

DISCUSSION

The ubiquitous function

$$\frac{\sin x}{x} \quad (84)$$

is sometimes called the diffraction function because it appears in the diffraction of waves at an aperture, but it appears in other applications as well. It should have a name of its own. It is proposed that it be called the sine quotient function.

The source density is a scalar while the vortex density is a vector. The sine quotient function can be used for the interpolation of the source density or either of the components of vortex density. Then the potential gradient of the pattern function is used directly with source density in the computation of radial flow, or in a vector product with either component of vortex density in the computation of circular flow.

A technique for the computation of ideal flow over arbitrary bodies has been invented by Hess and Smith^{4,5}. The same technique has been adopted for computation of flow over ships by Dawson and Dean^{6,7}. In this technique the surface of the body is subdivided by a grid, then each panel of the grid is projected onto a plane which spans the four

corners of the panel. It is assumed that the quadrilateral projection has a uniform source density. Each plane quadrilateral is assembled from plane triangles, for which the potential gradient is given by analytical expressions.

In the case of an infinite plane, uniform quadrilaterals would be reduced to rectangular plates. The potential gradient of a square plate is compared with the potential gradient of a source pattern in the following table.

Potential Gradients
 $a = b = 0.5$ $x = y = 0$

z	<i>Inverse Square</i>	<i>Plate</i>	<i>Pattern</i>
0	∞	2π	2π
0.0625	256	5.58063622	5.41516931
0.125	64	4.90438059	4.68192879
0.25	16	3.70918087	3.53400628
0.5	4	2.09439510	2.09491828
1.0	1	0.80543168	0.86196571
2.0	0.25	0.23543002	0.24793473
3.0	0.11111111	0.10812127	0.11106356
4.0	0.0625	0.06154089	0.06249869
5.0	0.04	0.03960461	0.03999996
6.0	0.02777777	0.02758643	0.02777777
7.0	0.02040816	0.02030466	0.02040816
8.0	0.015625	0.01556424	0.01562500

The entries in the table were computed with the aid of RPLTG, RPTNG, and RPTNX. The table shows that the inverse square law gives accurate gradients for $z > 4$.

For rectangular plates the field point must be kept away from the edges, where the gradient is infinite, whereas for pattern functions there is no restriction on the location of the field point.

CONCLUSION

Pattern functions which are based upon the sine quotient function are useful for the computation of the potential and the potential gradient of source distributions in the plane. Directly over a grid point the potential gradient has a direction which passes through the center of the pattern, but has a magnitude which is shielded from the full inverse square law.

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